Spectra With Positive Elementary Symmetric Functions

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ABSTRACT

It is well known that if all the elementary symmetric functions of the eigenvalues of an n-by-n matrix are positive, then all its eigenvalues lie in the region of the complex plane $\{z : -\pi + \pi/n < \arg z < \pi - \pi/n\}$. Let A denote an n-by-n matrix with all diagonal entries nonzero and for which the length of the longest cycle in its directed graph is $k, 2 \leqslant k \leqslant n$. If, in addition, all the cycles in the directed graph of $-A$ are signed negatively, then the elementary symmetric functions of the eigenvalues

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of A are positive and we ask whether its eigenvalues lie in the region $\{z : -\pi + \pi/k < \arg z < \pi - \pi/k\}$. This is known to be true when $k = 2$ (sign stability), and we prove it here for $k = n - 1$. We provide a counterexample for the case $k = n - 3$ and discuss related questions for more general classes of matrices with restricted length of the longest cycle.

1. INTRODUCTION

Let $A \in M_n(\mathscr{R})$, let $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ denote the spectrum of A, and let

$$\sigma_i = \sum_{1 \leqslant t_1 < \cdots < t_i \leqslant n} \prod_{l=1}^{i} \lambda_{t_l}, \qquad i = 1, 2, \dots, n,$$

be the ith elementary symmetric function of the eigenvalues. Of course, σ_i is also the sum of all the i-by-i principal minors of A, and the characteristic polynomial of A is

$$f(z) = z^n - \sigma_1 z^{n-1} + \sigma_2 z^{n-2} - \cdots + (-1)^{n-1}\sigma_{n-1} z + (-1)^n \sigma_n.$$

For a positive integer $k \geqslant 2$, let

$$W_k = \left\{ z \in \mathscr{C} : -\pi + \frac{\pi}{k} < \arg z < \pi - \frac{\pi}{k} \right\}.$$

In case $k = 1$, we define W_1 to be the positive real line. If $\sigma_i > 0$ for all $i = 1, 2, \dots, n$, it is known [14, 12, 3] that

$$\sigma(A) \subset W_n.$$

This approach was taken in [12] to establish the well-known exclusion region for the eigenvalues of *P-matrices*, i.e., matrices all of whose principal minors are positive. Analogous results about the location of the eigenvalues of nonnegative matrices were refined in [13], [8], and [9] as part of an effort to determine all possible eigenvalues of a nonnegative matrix with a specified graph. It is natural to ask whether a similar graph-theoretic refinement is possible for eigenvalue location results for P-matrices or for matrices with all $\sigma_i > 0$. For example, if the length of the longest cycle in the directed graph of A is less than n (this was the theme in [13] and [8]), the coefficients of the characteristic polynomial not only alternate in sign, but also have additional relations among them. As a result, there may be further restrictions on the location of the eigenvalues. Results concerning the location of eigenvalues of P-matrices under additional assumptions that are not graph-theoretic may be found in such papers as [3–6].

Before we proceed, let us recall some definitions. The *directed graph*, $D(A)$, of $A = (a_{ij}) \in M_n(\mathscr{R})$ consists of a set of vertices $\{1, 2, \ldots, n\}$ and a set of directed edges, with edge (i, j) connecting vertex i to vertex j if and only if $a_{ij} \neq 0$. $D(A)$ is *strongly connected* if for any pair of vertices i, j there exists a path in $D(A)$ from i to j. A *cycle of length* k in $D(A)$ is a set of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$, with the vertices $i_1, i_2, \ldots, i_k$ distinct. The nonzero diagonal entries of A correspond to cycles of length 1 in $D(A)$. The *signed digraph* of A, $\mathrm{SD}(A)$, is obtained from $D(A)$ by attaching to every edge (i, j) the sign of a_{ij}. We define the sign of the above cycle on the vertices $\{i_1, i_2, \ldots, i_k\}$ to be the sign of the product $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}$. We are interested in the matrices $A \in M_n(\mathscr{R})$ with nonzero diagonal entries such that the length of the longest cycle in $D(A)$ is no more than k; we denote the set of such matrices by $M_{n,k}$.

For matrices in $M_{n,k}$ we adopt the following notation. We write $A \in Q_{n,k}$ if the elementary symmetric functions of the eigenvalues of A are positive, $A \in P_{n,k}$ if A is a P-matrix, and $A \in S_{n,k}$ if all the cycles in $\mathrm{SD}(-A)$ are signed negatively. It follows from these definitions and the proof of Theorem 1.9 in [2] that

$$S_{n,k} \subset P_{n,k} \subset Q_{n,k} \subset M_{n,k}. \tag{1.1}$$

Notice that given $A \in S_{n,k}$, every matrix with the same sign pattern as A is also in $S_{n,k}$ and consequently a P-matrix. For that reason we refer to matrices in $S_{n,k}$ as *qualitative P-matrices* (see e.g. [7]).

For any $A \in Q_{n,k}$ we have that $\sigma(A) \subset W_n$. It is also clear that if $A \in Q_{n,1}$ then $\sigma(A) \subset W_1$. Furthermore, it was shown in [16] and [10] that $S_{n,2}$ is a subset of the *positive sign-stable matrices*, namely, matrices having all their eigenvalues in the open right half plane by virtue of their sign pattern and regardless of the magnitude of their entries. Hence, if $A \in S_{n,2}$, then $\sigma(A) \subset W_2$. Motivated by these results, we ask the following question:

$$\text{If} \quad A \in S_{n,k}, \quad \text{is} \quad \sigma(A) \subset W_k? \tag{1.2}$$

In [1] (see also Theorem 21.5.1 in [7]) it was shown that if $A \in M_{n,2}$, then $-A$ has a diagonal Lyapunov solution (and hence A is positive stable) if and only if A is a P-matrix. This suggests the following generalization of (1.2):

$$\text{If} \quad A \in P_{n,k}, \quad \text{is} \quad \sigma(A) \subset W_k? \tag{1.3}$$

It is natural to ask whether these questions can be further generalized to matrices in $Q_{n,k}$. However, the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & -3 & 6 \\ 0 & -6 & 2 \end{pmatrix} \in Q_{3,2}$$

has $-0.5 + i5.0744 \in \sigma(A)$, and therefore the assumption that A is a P-matrix cannot be relaxed.

Our approach, in order to investigate the questions that we posed, is to obtain conditions on the location of roots of polynomials (Section 2) and then apply them to the characteristic polynomials of matrices. In Section 3 we show that (1.2) has a positive answer for $k = n - 1$ (Theorem 3.3) and therefore for any $A \in S_{n,k}$, $k < n$, $\sigma(A) \subset W_{n-1}$. In Section 4 we provide a counterexample for (1.2), and thus (1.3), when $k = n - 3$, and discuss cases of matrices $S_{n,k}$ and $P_{n,k}$ whose eigenvalues lie in W_k. The questions for $P_{n,n-1}$, $S_{n,n-2}$, and $P_{n,n-2}$ remain open.

2. ON THE LOCATION OF ROOTS OF POLYNOMIALS

Let $f(z)$ be the real polynomial

$$f(z) = z^n - \sigma_1 z^{n-1} + \sigma_2 z^{n-2} - \cdots + (-1)^{n-1}\sigma_{n-1} z + (-1)^n \sigma_n \tag{2.1}$$

with *arbitrary* coefficients σ_i. For any fixed θ and a complex number $z = re^{i\theta}$ we write

$$f(re^{i\theta}) = U(r) + iV(r),$$

where

$$U(r) = r^n \cos n\theta - \sigma_1 r^{n-1} \cos(n-1)\theta + \cdots$$
$$+ (-1)^{n-1}\sigma_{n-1} r \cos\theta + (-1)^n \sigma_n \tag{2.2}$$

and

$$V(r) = r^n \sin n\theta - \sigma_1 r^{n-1} \sin(n-1)\theta + \cdots + (-1)^{n-1}\sigma_{n-1} r \sin\theta. \tag{2.3}$$

By $I_0^\infty(V(r)/U(r))$ we denote the Cauchy index of the real rational function $V(r)/U(r)$, namely, the difference between the number of jumps from $-\infty$ to $+\infty$ and the number of jumps from $+\infty$ to $-\infty$ of $V(r)/U(r)$, as r varies from 0 to $+\infty$. The following theorem specifies the number of roots λ of $f(z)$ such that $|\arg\lambda| < \theta$ (see Theorem 1 and Corollary 1 of [3] and also Section 41 of [14]).

THEOREM 2.1. *Given any fixed $\theta \in (0, \pi)$, if the real polynomial $f(z)$ in* (2.1) *has no root λ such that* $\arg\lambda = \theta$, *and if* $\cos n\theta \neq 0$, *then the number L of its roots λ such that* $|\arg\lambda| < \theta$ *is equal to*

$$L = I_0^\infty \frac{V(r)}{U(r)} + \frac{n\theta}{\pi} - \frac{1}{\pi}\arctan(\tan n\theta),$$

where $U(r)$ and $V(r)$ are given in (2.2) *and* (2.3) *respectively.*

Notice that, on letting $\theta = \pi - \pi/k$ in Theorem 2.1, our questions can be answered by computing the quantity L for characteristic polynomials. The following theorem provides a necessary and sufficient condition on the Cauchy index so that $L = n$, thus establishing that the roots of $f(z)$ lie in W_k.

THEOREM 2.2. *Let $2 \leqslant k \leqslant n$, and let $p \geqslant 1$ and $0 \leqslant m < k$ be integers such that $n = pk + m$. Let also $\theta = \pi - \pi/k$. Under the assumptions of Theorem* 2.1, *every root λ of $f(z)$ satisfies* $|\arg\lambda| < \theta$ *if and only if*

$$I_0^\infty \frac{V(r)}{U(r)} = \begin{cases} p & \text{if } m/k \leqslant \frac{1}{2}, \\ p+1 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.1 for $\theta = \pi - \pi/k$, each root λ of $f(z)$ satisfies $|\arg\lambda| < \theta$ if and only if $L = n$, that is,

$$I_0^\infty \frac{V(r)}{U(r)} + n - \frac{n}{k} - \frac{1}{\pi}\arctan\left[\tan\left((pk+m)\pi - p\pi - \frac{m\pi}{k}\right)\right] = n,$$

or equivalently,

$$I_0^\infty \frac{V(r)}{U(r)} - \frac{n}{k} - \frac{1}{\pi} \arctan\left[\tan\left(-\frac{m\pi}{k}\right)\right] = 0. \tag{2.4}$$

The proof is completed by observing that

$$\arctan\left[\tan\left(-\frac{m}{k}\pi\right)\right] = \begin{cases} -\dfrac{m}{k}\pi & \text{if } \dfrac{m}{k} \leqslant \dfrac{1}{2}, \\ \dfrac{k-m}{k}\pi & \text{otherwise.} \end{cases}$$ ■

COROLLARY 2.3. *Let $A \in M_n(\mathscr{R})$ have positive diagonal entries and characteristic polynomial $f(z)$ as in (2.1). Let $\theta = \pi - \pi/k$; suppose that* $\cos n\theta \neq 0$ *and that $n = pk + m$ as in Theorem 2.2. Then $\sigma(A) \subset W_k$ if and only if*

$$I_0^\infty \frac{V(r)}{U(r)} = \begin{cases} p & \text{if } m/k \leqslant \frac{1}{2} \\ p+1 & \text{otherwise,} \end{cases}$$

where $U(r)$ and $V(r)$ are given in (2.2) and (2.3) respectively.

Proof. Let $\theta = \pi - \pi/k$. First assume that no $\lambda \in \sigma(A)$ has $\arg\lambda = \theta$. Then the corollary follows from Theorem 2.2 applied to the characteristic polynomial of A. However, this result is true without the assumption on $\arg\lambda$, as the following argument shows. If $\lambda \in \sigma(A)$ has $\arg\lambda = \theta$, then for some $\epsilon > 0$ small enough the matrix $A_\epsilon = A - \epsilon I \in M_n(\mathscr{R})$ has positive diagonal entries. But then $\lambda - \epsilon I \in \sigma(A_\epsilon)$ and $\theta < \arg(\lambda - \epsilon I) < \pi$, which is a contradiction. ■

3. THE CASE $k = n - 1$

In this section we will obtain necessary and sufficient conditions on the elementary symmetric functions of the eigenvalues of a matrix $A \in P_{n,n-1}$ so that $\sigma(A) \subset W_{n-1}$. Since this result is true for $n \leqslant 3$, let us assume that $n > 3$. Let now $f(z)$ in (2.1) be the characteristic polynomial of A, and let $U(r)$ and $V(r)$ be the real and the imaginary part of $f(re^{i\theta})$ as defined in (2.2) and (2.3) respectively. As $A \in P_{n,n-1}$, all $\sigma_i > 0$. We will evaluate explicitly the Cauchy index of $V(r)/U(r)$. When $k = n - 1$, then $p = m = 1$

and $\cos n\theta \neq 0$. Therefore, by Corollary 2.3, the result is true if and only if

$$I_0^\infty \frac{V(r)}{U(r)} = 1. \tag{3.1}$$

In order to investigate (3.1) for $A \in P_{n,n-1}$, we need to compute the number and the relative location of the real positive roots of $V(r)$ and $U(r)$. Let now $\theta = \pi - \pi/(n-1)$. First observe that, for $l = 0, 1, \ldots, n-1$,

$$\sin(n-l)\theta = \begin{cases} \sin \dfrac{n-l}{n-1}\pi & \text{if } \ n-l \ \text{ is odd,} \\ -\sin \dfrac{n-l}{n-1}\pi & \text{otherwise.} \end{cases}$$

But

$$\sin \frac{n-l}{n-1}\pi$$

is negative when $l = 0$, zero when $l = 1$, and positive for $l = 2, 3, \ldots, n-1$. Hence the coefficients of $V(r)$ in (2.2) alternate in sign once, and, by Descartes's rule of signs, $V(r)$ has exactly one real positive root, denoted by r_v. Similarly, for $l = 0, 1, \ldots, n-1$,

$$\cos(n-l)\theta = \begin{cases} -\cos \dfrac{n-l}{n-1}\pi & \text{if } \ n-l \ \text{ is odd,} \\ \cos \dfrac{n-l}{n-1}\pi & \text{otherwise,} \end{cases}$$

and since

$$\cos \frac{n-l}{n-1}\pi$$

switches sign once as l goes from 0 to $n-1$, $U(r)$ also has a unique real positive root, denoted by r_u.

Observe now that when $r > 0$ is close to 0,

$$(-1)^n U(r) > 0 \quad \text{and} \quad (-1)^{n-1} V(r) > 0.$$

Consequently, in order for $V(r)/U(r)$ to have a jump from $-\infty$ to $+\infty$ at $r = r_u$, we must have that $r_u < r_v$, or equivalently that

$$(-1)^{n-1} U(r_v) > 0. \tag{3.2}$$

In what follows we will compute $U(r_v)$ and then use the inequality in (3.2) as a necessary and sufficient condition for any eigenvalue λ of $A \in P_{n,n-1}$ to satisfy $|\arg \lambda| < \theta$.

Observe that r can be factored out from $V(r)$. Then, solving the equation $V(r_v)/r_v = 0$ for r_v^{n-1}, since $\sin(n-1)\theta = 0$, we obtain

$$\begin{aligned} r_v^{n-1} = & -\sigma_2 r_v^{n-3} \frac{\sin(n-2)\theta}{\sin n\theta} + \sigma_3 r_v^{n-4} \frac{\sin(n-3)\theta}{\sin n\theta} \\ & - \cdots + (-1)^n \sigma_{n-1} \frac{\sin\theta}{\sin n\theta}. \end{aligned} \tag{3.3}$$

Notice that

$$\sin n\theta = (-1)^n \sin\theta, \qquad \cos(n-1)\theta = (-1)^n,$$

and that, for $l = 2, 3, \ldots, n$,

$$\cos(n-l)\theta - \frac{\cos n\theta \sin(n-l)\theta}{\sin\theta} = \frac{\sin l\theta}{\sin n\theta}.$$

Using the above relations, when we substitute r_v^{n-1} from (3.3) in the expression for $U(r_v)$, we obtain

$$U(r_v) = \sum_{l=2}^{n} (\gamma_l \sigma_l + \delta_l \sigma_1 \sigma_{l-1}) r_v^{n-l}, \tag{3.4}$$

where, for $l = 2, 3, \ldots, n$,

$$\gamma_l = (-1)^{n-l} \frac{\sin l\theta}{\sin\theta} \quad \text{and} \quad \delta_l = (-1)^{l-1} \frac{\sin(n-l+1)\theta}{\sin\theta}. \tag{3.5}$$

We will now investigate the signs of γ_l and δ_l. We have that

$$\sin l\theta = \begin{cases} \sin \dfrac{l}{n-1}\pi > 0 & \text{if } l\ (\leqslant n-2) \text{ is odd,} \\ -\sin \dfrac{l}{n-1}\pi < 0 & \text{if } l\ (\leqslant n-2) \text{ is even,} \\ 0 & \text{if } \quad l = n-1, \\ (-1)^n \sin\theta & \text{if } \quad l = n. \end{cases} \tag{3.6}$$

Consequently,

$$(-1)^{n-1}\gamma_l \geqslant 0 \text{ and } (-1)^{n-1}\delta_l \geqslant 0 \qquad \text{for} \quad l = 2, 3, \dots, n-1, \tag{3.7}$$

and, in particular,

$$\gamma_{n-1} = \delta_2 = 0, \qquad (-1)^{n-1}\delta_n > 0, \quad \text{and} \quad (-1)^n \gamma_n > 0. \tag{3.8}$$

The coefficients of σ_n and $\sigma_1\sigma_{n-1}$ in (3.4) respectively, satisfy

$$|\gamma_n| = 1 \quad \text{and} \quad |\delta_n| = 1. \tag{3.9}$$

Since we have no estimate on the value of r_v, it will be very difficult to verify the condition in (3.2) for the expressions in (3.4) and (3.5). For that purpose we will now assume that $r_v = 1$. This assumption is made without loss of generality, since we can scale our matrix A by a factor r_v^{-1}. The multilinearity of the elementary symmetric functions of the eigenvalues of A implies that the imaginary part of the characteristic polynomial of the scaled matrix has a unique positive root equaling 1. Of course this assumption imposes an important and useful condition on the elementary symmetric functions. If $r_v = 1$, from $V(1) = 0$ we obtain

$$-\sigma_2 \frac{\sin(n-2)\theta}{\sin n\theta} + \sigma_3 \frac{\sin(n-3)\theta}{\sin n\theta} - \cdots + (-1)^n \sigma_{n-1} \frac{\sin\theta}{\sin n\theta} = 1,$$

which can be rewritten as

$$\sum_{l=2}^{n-1} \alpha_l \sigma_l = 1, \qquad \text{where} \quad \alpha_l = \frac{\sin \dfrac{n-l}{n-1}\pi}{\sin \dfrac{1}{n-1}\pi} > 1. \tag{3.10}$$

We can state now the following result.

THEOREM 3.1. *Given $A \in P_{n,n-1}$, $n > 3$, let σ_i, $i = 1, 2, \dots, n$, denote the elementary symmetric functions of the eigenvalues of A. Let also α_l, γ_l, and δ_l be as defined in* (3.10) *and* (3.5). *Then, under the condition that*

$$\sum_{l=2}^{n-1} \alpha_l \sigma_l = 1,$$

$\sigma(A) \subset W_{n-1}$ *if and only if*

$$\sigma_n < \sigma_1 \sigma_{n-1} + \sum_{l=2}^{n-2} |\gamma_l| \sigma_l + \sum_{l=3}^{n-1} |\delta_l| \sigma_1 \sigma_{l-1}.$$

Proof. Recall that (3.2) is a necessary and sufficient condition for $\sigma(A) \subset W_{n-1}$. The theorem then follows from the normalization $r_v = 1$ and from (3.4), (3.7), and (3.8). ∎

We will now apply Theorem 3.1 to matrices in $S_{n,n-1}$ and show that the answer to (1.2) is affirmative for $k = n - 1$. First, we need the following lemma. For more details on the concepts used in the proof of this lemma see [15, Section 2].

LEMMA 3.2. *Let $A \in S_{n,k}$, $n \geqslant 3$, $k \leqslant n - 1$, and let σ_i, $i = 1, 2, \dots, n$, denote the elementary symmetric functions of the eigenvalues of A. Then,*

$$\sigma_n < \sigma_1 \sigma_{n-1} + \sigma_2 \sigma_{n-2} + \cdots + \sigma_{\lfloor n/2 \rfloor} \sigma_{\lceil n/2 \rceil}. \tag{3.11}$$

Proof. Let $A = (a_{ij})$ and recall that

$$\det A = \sigma_n = \sum_{\phi} (\operatorname{sgn} \phi) a_{1,\phi(1)} a_{2,\phi(2)} \cdots a_{n,\phi(n)}, \tag{3.12}$$

where the summation is over all permutations ϕ of $\{1, 2, \dots, n\}$. Observe that each summand in (3.12) is nonnegative, since $A \in S_{n,k}$. Similarly, since every cycle of a principal submatrix of A is a cycle of A, and since σ_i is the sum of all $i \times i$ principal minors of A, each summand in the expansion of σ_i is also nonnegative for all $i = 1, 2, \dots, n-1$. Let now

$$\beta_\phi = (\operatorname{sgn} \phi) a_{1,\phi(1)} a_{2,\phi(2)} \cdots a_{n,\phi(n)} > 0$$

be any one of the nonzero summands in (3.12) corresponding to the permutation ϕ. Consider then the decomposition of ϕ into disjoint cycles, and let $q \leqslant k$ be the cardinality of any one of its cyclic factors. Observe then that β_ϕ appears in $\sigma_q \sigma_{n-q}$ on the right-hand side of (3.11). Also, the nonzero diagonal entries of A ensure that the inequality is strict, completing the proof of the lemma. ■

THEOREM 3.3. *If $A \in S_{n,n-1}$, then $\sigma(A) \subset W_{n-1}$.*

Proof. If $n \leqslant 3$, then, as we remarked in the introduction, A is a positive sign-stable matrix and the result is true. Suppose that $n > 3$ and let $\theta = \pi - \pi/(n-1)$. Let γ_l and δ_l be as defined in (3.5). By (3.6) and a similar analysis for $\sin(n-l+1)\theta$, γ_l and δ_l satisfy

$$|\gamma_l| > 1 \qquad \text{for} \quad l = 2, 3, \dots, n-2, \tag{3.13}$$

and

$$|\delta_l| > 1 \qquad \text{for} \quad l = 3, 4, \dots, n-1. \tag{3.14}$$

Suppose now that A has been normalized so that (3.10) holds. Since the coefficients α_l in (3.10) also satisfy $\alpha_l > 1$, we must have that

$$\sigma_l < 1 \qquad \text{for} \quad l = 2, 3, \dots, n-1. \tag{3.15}$$

Therefore, by Lemma 3.2, the relations in (3.15), (3.13), (3.14) in that order, and the fact that $\sigma_l > 0$, for all $l = 1, 2, \dots, n$, we obtain

$$\begin{aligned}
\sigma_n &< \sigma_1 \sigma_{n-1} + \sigma_2 \sigma_{n-2} + \cdots + \sigma_{\lfloor n/2 \rfloor} \sigma_{\lceil n/2 \rceil} \\
&< \sigma_1 \sigma_{n-1} + \sigma_2 + \sigma_3 + \cdots + \sigma_{\lfloor n/2 \rfloor} + \cdots + \sigma_{n-2} \\
&< \sigma_1 \sigma_{n-1} + \sum_{l=2}^{n-2} |\gamma_l| \sigma_l + \sum_{l=3}^{n-1} |\delta_l| \sigma_1 \sigma_{l-1}.
\end{aligned}$$

The above inequality, combined with Theorem 3.1, concludes the proof of the theorem. ■

4. COUNTEREXAMPLE AND SPECIAL CASES

Consider the matrix

$$A = \begin{pmatrix} 10^{-2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 10^{-2} & 1 & 0 & 0 & 0 \\ 10 & 0 & 10^{-2} & 10^2 & 0 & 0 \\ 0 & 0 & -1 & 10^{-2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 10^{-2} & 1 \\ 0 & 0 & 0 & 10 & 0 & 10^{-2} \end{pmatrix} \in S_{6,3}.$$

Since A has an eigenvalue with argument $132.8° > 120°$, the answer to (1.2) is not in the affirmative for all n and k. This example, which is due to M. Bakonyi, can be generalized to show that (1.2) is not true for $S_{n,n-3}$ [by considering an appropriate matrix A with $D(A)$ consisting of a cycle of length $n-3$ and a disjoint cycle of length 3, connected by a cycle of length 2]. At this point, the smallest unknown case for (1.2) is when $n = 5$ and $k = 3$.

Let $A \in S_{5,3}$. In order to apply Corollary 2.3 to A, we compute $U(r)$ and $V(r)$ as in Section 3, with $\theta = \pi - \pi/3$, and we obtain

$$U(r) = -\tfrac{1}{2}r^5 + \tfrac{1}{2}\sigma_1 r^4 + \sigma_2 r^3 + \tfrac{1}{2}\sigma_3 r^2 - \tfrac{1}{2}\sigma_4 r - \sigma_5$$

and

$$V(r) = -\frac{\sqrt{3}}{2}r^5 - \frac{\sqrt{3}}{2}\sigma_1 r^4 + \frac{\sqrt{3}}{2}\sigma_3 r^2 + \frac{\sqrt{3}}{2}\sigma_4 r.$$

By Descartes's rule of signs, $V(r)$ has a unique positive root r_v, while $U(r)$ has either two positive roots or none at all. By Corollary 2.3, all $\lambda \in \sigma(A)$ satisfy $|\arg \lambda| < \theta$ if and only if $I_0^\infty V(r)/U(r) = 2$. Notice that, by considering the signs of $U(r)$ and $V(r)$ at the endpoints of $[0, \infty)$, this condition on the Cauchy index holds if and only if $U(r_v) > 0$. Normalizing so that $r_v = 1$, as we did in Section 3, and requiring that $U(1) > 0$, we have that $|\arg \lambda| < \theta$ for all $\lambda \in \sigma(A)$ if and only if

$$\sigma_4 + \sigma_5 < \sigma_1 + \sigma_2 \tag{4.1}$$

under the condition that

$$1 + \sigma_1 = \sigma_3 + \sigma_4. \tag{4.2}$$

It is worth noting at this point that for matrices in $S_{5,3}$ the proof of Lemma 3.2 in fact shows

$$\sigma_5 < \sigma_3\sigma_2. \tag{4.3}$$

In the following result for a matrix $A \in S_{5,3}$ we will show, under the additional assumption that $D(A)$ has no cycles of length 2, that (4.2) implies (4.1).

PROPOSITION 4.1. *Let $A \in S_{5,3}$ such that $D(A)$ has no cycles of length 2. Then $\sigma(A) \subset W_3$.*

Proof. According to the discussion preceding this proposition, we will assume that the elementary symmetric functions σ_i of the eigenvalues of A satisfy (4.2), and we will show that (4.1) holds. First, observe that

$$\sigma_1\sigma_5 < \sigma_2\sigma_4. \tag{4.4}$$

This inequality follows by expressing σ_1, σ_2, σ_4, and σ_5 as sums of principal minors of A and comparing the summands of their expansions as we did in the proof of Lemma 3.2. It is important to notice that, since the digraph of A is assumed to have no cycles of length 2, σ_2 is equal to the second elementary function of the diagonal entries of A. Similarly, we can show that

$$\sigma_4 < \sigma_1\sigma_3. \tag{4.5}$$

If $\sigma_3 \leqslant 1$, then, by (4.5) and (4.2), we have $\sigma_4 < \sigma_1$ and $\sigma_4 \geqslant \sigma_1$, a contradiction. Therefore $\sigma_3 > 1$. Then, by (4.2) and (4.4), $\sigma_4 < \sigma_1$ and $\sigma_5 < \sigma_2$, which completes the proof of the proposition. ∎

Notice that if $A \in S_{5,3}$ is such that $D(A)$ has no cycles of length 2, then A is signature-similar to a nonnegative matrix. Thus A has a positive Perron eigenvalue, $r = \rho(A)$. By Theorem 1 in [13], if $\lambda = \mu + i\nu \in \sigma(A)$, then

$$\mu + |\nu|\tan\frac{\pi}{3} \leqslant \rho(A).$$

This result, along with Proposition 4.1, restricts the eigenvalues of such a matrix to the shaded double-wedge region in Figure 1.

We conclude our discussion with a subclass of $P_{n,n-1}$ matrices whose eigenvalues lie in W_{n-1}. Consider a P-matrix A with the following zero-non-

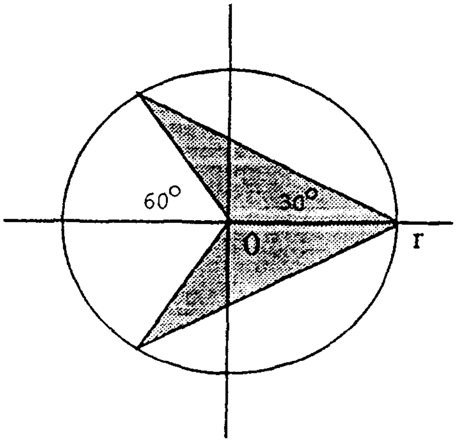

FIG. 1.

zero pattern:

$$A = \begin{pmatrix} \times & \times & 0 & 0 \\ 0 & \times & \times & 0 \\ \times & \times & \times & \times \\ 0 & \times & 0 & \times \end{pmatrix}.$$

According to the determinantal identity in [11], since $D(A)$ has no Hamilton cycle, the determinant of A can be expressed entirely in terms of principal minors of A of order less than n as follows:

$$\det A = \det A[1,2,3] \det A[4] + \det A[2,3,4] \det A[1]$$
$$- \det A[2,3] \det A[1] \det A[4].$$

From this identity, as A is a P-matrix, we obtain

$$\det A < \det A[1,2,3] \det A[4] + \det A[2,3,4] \det A[1]$$

and therefore $\sigma_4 < \sigma_1 \sigma_3$. Consequently, the inequality (3.11) is satisfied by A, and, as in the proof of Theorem 3.3, $\sigma(A) \subset W_{n-1}$. In the terminology of [11], this result can be generalized to any $A \in P_{n,k}$ with the property that its maximal cycle cover consists of one or two partitions, of cardinality 2. For any such matrix A, $\sigma(A) \subset W_{n-1}$.

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